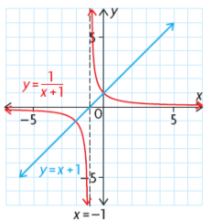
Unit 2 Summary - Transformations and Rational Functions

- The polynomial function $y = a(k(x - d))^n + c$ can be graphed by applying transformations to the graph of the parent function $y = x^n$, where $n \in \mathbf{N}$. Each point (x, y) on the graph of the parent function changes to $\left(\frac{x}{k} + d, ay + c\right)$.
- When using transformations to graph a function in the fewest steps, you can apply a and k together, and then c and d together.
- In $y = a(k(x - d))^n + c$,
  - the value of a represents a vertical stretch/compression and possibly a vertical reflection
  - the value of k represents a horizontal stretch/compression and possibly a horizontal reflection
  - the value of d represents a horizontal translation
  - the value of c represents a vertical translation

• You can use key characteristics of the graph of a linear or quadratic function to graph the related reciprocal function.
All the y-coordinates of a reciprocal function are the reciprocals of the y-coordinates of the original function.
• The graph of a reciprocal function has a vertical asymptote at each zero of the original function.
• A reciprocal function will always have y=0 as a horizontal asymptote if the original function is linear or quadratic.
• A reciprocal function has the same positive/negative intervals as the original function.

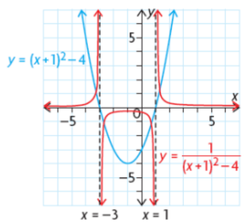
- Intervals of increase on the original function are intervals of decrease on the reciprocal function. Intervals of decrease on the original function are intervals of increase on the reciprocal function.
- If the range of the original function includes 1 and/or -1, the reciprocal function will intersect the original function at a point (or points) where the y-coordinate is 1 or -1.
- If the original function has a local minimum point, the reciprocal function will have a local maximum point at the same x-value (and vice versa).

A linear function and its reciprocal



Both functions are negative when $x \in (-\infty, -1)$ and positive when $x \in (-1, \infty)$. The original function is increasing when $x \in (-\infty, \infty)$. The reciprocal function is decreasing when $x \in (-\infty, -1)$ or $(-1, \infty)$.

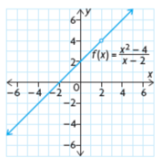
A quadratic function and its reciprocal



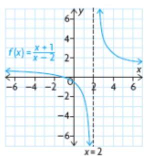
Both functions are negative when $x \in (-3, 1)$ and positive when $x \in (-\infty, -3)$ or $(1, \infty)$. The original function is decreasing when $x \in (-\infty, -1)$ and increasing when $x \in (-1, \infty)$. The reciprocal function is increasing when $x \in (-\infty, -3)$ or $(-3, -1)$ and decreasing when $x \in (-1, 1)$ or $(1, \infty)$.

- The quotient of two polynomial functions results in a rational function which often has one or more discontinuities.
- The breaks or discontinuities in a rational function occur where the function is undefined. The function is undefined at values where the denominator is equal to zero. As a result, these values must be restricted from the domain of the function.
- The values that must be restricted from the domain of a rational function result in key characteristics that define the shape of the graph. These characteristics include a combination of vertical asymptotes (also called infinite discontinuities) and holes (also called point discontinuities).
- The end behaviours of many rational functions are determined by either horizontal asymptotes or oblique asymptotes.

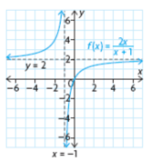
- A rational function, $f(x) = \frac{p(x)}{q(x)}$, has a hole at $x = a$ if $\frac{p(a)}{q(a)} = \frac{0}{0}$. This occurs when $p(x)$ and $q(x)$ contain a common factor of $(x - a)$. For example, $f(x) = \frac{x^2 - 4}{x - 2}$ has the common factor of $(x - 2)$ in the numerator and the denominator. This results in a hole in the graph of $f(x)$ at $x = 2$.



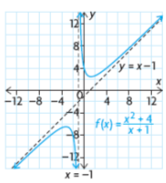
- A rational function, $f(x) = \frac{p(x)}{q(x)}$, has a vertical asymptote at $x = a$ if $\frac{p(a)}{q(a)} = \frac{p(a)}{0}$. For example, $f(x) = \frac{x + 1}{x - 2}$ has a vertical asymptote at $x = 2$.



- A rational function, $f(x) = \frac{p(x)}{q(x)}$, has a horizontal asymptote only when the degree of $p(x)$ is less than or equal to the degree of $q(x)$. For example, $f(x) = \frac{2x}{x + 1}$ has a horizontal asymptote at $y = 2$.

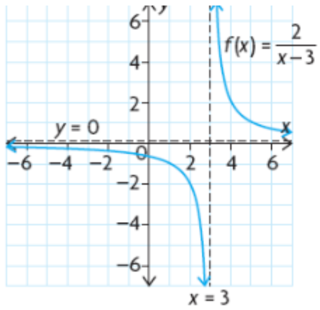


- A rational function, $f(x) = \frac{p(x)}{q(x)}$, has an oblique (slant) asymptote only when the degree of $p(x)$ is greater than the degree of $q(x)$ by exactly 1. For example, $f(x) = \frac{x^2 + 4}{x + 1}$ has an oblique asymptote.

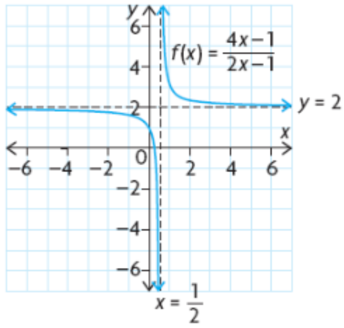


- The graphs of most rational functions of the form $f(x) = \frac{b}{cx + d}$ and $f(x) = \frac{ax + b}{cx + d}$ have both a vertical asymptote and a horizontal asymptote.
- You can determine the equation of the vertical asymptote directly from the equation of the function by finding the zero of the denominator.
- You can determine the equation of the horizontal asymptote directly from the equation of the function by examining the ratio of the leading coefficients in the numerator and the denominator. This gives you the end behaviours of the function.
- To sketch the graph of a rational function, you can use the domain, intercepts, equations of asymptotes, and positive/negative intervals.

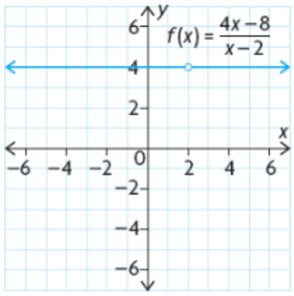
- Rational functions of the form $f(x) = \frac{b}{cx + d}$ have a vertical asymptote defined by $x = -\frac{d}{c}$ and a horizontal asymptote defined by $y = 0$. For example, see the graph of $f(x) = \frac{2}{x - 3}$.



- Most rational functions of the form $f(x) = \frac{ax + b}{cx + d}$ have a vertical asymptote defined by $x = -\frac{d}{c}$ and a horizontal asymptote defined by $y = \frac{a}{c}$. For example, see the graph of $f(x) = \frac{4x - 1}{2x - 1}$.



The exception occurs when the numerator and the denominator both contain a common linear factor. This results in a graph of a horizontal line that has a hole where the zero of the common factor occurs. As a result, the graph has no asymptotes. For example, see the graph of $f(x) = \frac{4x - 8}{x - 2} = \frac{4(x - 2)}{(x - 2)}$.



- You can solve a rational equation algebraically by multiplying each term in the equation by the lowest common denominator and solving the resulting polynomial equation.
- The root of the equation $\frac{ax + b}{cx + d} = 0$ is the zero (x-intercept) of the function $f(x) = \frac{ax + b}{cx + d}$.
- You can use graphing technology to solve a rational equation or verify the solution. Determine the zeros of the corresponding rational function, or determine the intersection of two functions.

- The zeros of a rational function are the zeros of the function in the numerator.
- Reciprocal functions do not have zeros. All functions of the form $f(x) = \frac{1}{g(x)}$ have the x-axis as a horizontal asymptote. They do not intersect the x-axis.
- When solving contextual problems, it is important to check for inadmissible solutions that are outside the domain determined by the context.
- When using a graphing calculator to determine a zero or intersection point, you can avoid inadmissible roots by matching the window settings to the domain of the function in the context of the problem.